International Journal on Emerging Technologies 10(4): 201-211(2019)

ISSN No. (Print): 0975-8364
ISSN No. (Online): 2249-3255

Approximate Solution of Ordinary Differential Equations via Hybrid Block Approach

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(Received 29 August 2019, Revised 25 October 2019, Accepted 04 November 2019)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: In this paper, a hybrid block approach (six-step) is proposed to find the approximate solution for the fifth order differential equation, instead of reducing it to the lowest order systems of equations. Three numerical test problems have done to test the efficiency and these are compared with the results of ODE45, exact solutions and results in the recent literature. The construction of this method has made with collocation and interpolation at both the step and off step points. The validity of the paper is analyzed through convergence, stability and numerical test problems.

Keywords: Block method, collocation, convergence, interpolation, hybrid, stability.

I. INTRODUCTION

The mathematical modeling constructs various real-world physical problems to mathematical problems with the application of ordinary differential equations involving initial conditions in engineering and technology [6-9]. Various analytical approaches unlike Laplace transformation, power series method are tedious to determined solutions of initial value problems.

In this paper, the numerical approach has been designed to solve for the higher order differential equations of the form

$$z^{(w)} = g(t, z, z^{(1)}, z^{(2)}, ..., z^{(w-1)}),\ z^{(\eta)}(t_0) = z_\eta,$$
$$\eta = 0(1)(w-1),\ t \in [t_0,\ a] \tag{1}$$

These types of problems have many applications in science and engineering, especially in mechanical systems, control theory, and celestial mechanics. In recent years many researchers have been solve Eqn. (1) by reducing to systems of first order differential equations with applying of any suitable numerical approach [10], [16-18, 24]. Some researchers to solve Eqn. (1) they have applied direct approach, like various types of block methods with less order [4, 5, 10, 12, 20, 22, 26, 27], application of haar wavelet collocation method [1], finite difference method [9], a multiderivative collocation method [13], an order seven continuous explicit method [2], different type of hybrid block method with lower order [3, 11], a P-stable method [14, 15], an efficient zero-stable numerical approach [21], linear multistep method [25] and also numbers of researchers have an interest in development of direct integration approaches [19, 23]. But the solutions to these reduction approaches, direct methods and integration approaches also have less accuracy. That drawback has been observed when dealing with a higher order IVPs. A six-step hybrid block is developed by combing the of a family of collocation and interpolation at the step(grid) and off-step(off-grid) points to improve the accuracy and to test the efficiency of the proposed method, respectively.

The power series has been taken as the basis function for solving the approximate solution of linear and nonlinear higher order IVPs arising in mathematical modeling in different physical applications. Particularly for in Eqn. (1)

$$z^{(5)} = g(t, z,\ z', z'', z''',\ z^{(4)}),$$
$$z^{(\eta)}(t_0) = z_\eta,\ \eta = 0(1)4,\ t \in [t_0,\ a] \tag{2}$$

Specially Eqn. (2) has many applications in civil, mechanical, applied science and aerospace engineering. In recent years Eqn. (2) is solved by different researchers [1, 2, 13, 23].

This paper is summarized as follows: Section II contains the construction six step hybrid block method. The properties of the block method are discussed in Section III. Three numerical test problems and results are discussed in Section IV. The conclusion is drawn in Section V.

II. CONSTRUCTION OF THE METHOD

In this part, the construction of a six-step hybrid block method with six off-step(off-grid) points $t_{l+1/2}, t_{l+3/2},\ t_{l+5/2},\ t_{l+7/2},\ t_{l+9/2}, t_{l+11/2}$ have been discussed for solving Eqn. (2).

Let us consider the power series of the form

$$z(t) = \sum_{k=0}^{\alpha+s-1} d_k t^k \tag{3}$$

is used as a basis function to get the approximate solution of Eqn. (2). The fifth derivative of Eqn. (3) is given as

$$z^{(5)} = \sum_{k=0}^{\alpha+s-1} k(k-1)(k-2)(k-3)(k-4) d_k t^{k-5} \tag{4}$$

Eqn. (5) has been obtained by substituting Eqn. (4) into Eqn. (2)

$$z^{(5)} = g(t, z,\ z^{(1)}, z^{(2)},\ z^{(3)},\ z^{(4)})$$
$$= \sum_{k=0}^{\alpha+s-1} k(k-1)(k-2)(k-3)(k-4) d_k t^{k-5} \tag{5}$$

where α is the number of interpolating points (which is the same as the order of the given differential equations), s is the number of collocation points and d_k is the parameters to be determined.

The non-linear system of equations in Eqn. (6) and (7) are obtained by interpolating at the points $t_{l+\eta}$, $\eta = 0(1)4$ and collocating at each point

t_{l+r}, $r = 0\left(\frac{1}{2}\right)6$ from Eqn. (3) and (5) respectively.

$$z(t_{l+\eta}) = \sum_{k=0}^{17} d_k \left(t_{l+\eta}\right)^k \qquad (6)$$

$$g_{l+r} = g\left(t_{l+r},\ z_{l+r}, z_{l+r}^{(1)}, z_{l+r}^{(2)},\ z_{l+r}^{(3)}, z_{l+r}^{(4)}\right)$$
$$= \sum_{k=0}^{17} k(k-1)(k-2)(k-3)(k-4) d_k \left(t_{l+r}\right)^{k-5} \qquad (7)$$

The system in Eqns. (6) and (7), can be written in matrix form as

$$DX = C \qquad (8)$$

where

$$D = \begin{pmatrix} 1 & t_l & \cdots & t_l^5 & t_l^6 & \cdots & t_l^{17} \\ 1 & t_l+h & \cdots & (t_l+h)^5 & (t_l+h)^6 & \cdots & (t_l+h)^{17} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & t_l+4h & \cdots & (t_l+4h)^5 & (t_l+4h)^6 & \cdots & (t_l+4h)^{17} \\ 0 & \cdots & 0 & 120 & 720\ t_l & \cdots & 742560\ \ t_l^{12} \\ 0 & \cdots & 0 & 120 & 720\left(t_l+\frac{h}{2}\right) & \cdots & 742560\ \left(t_l+\frac{h}{2}\right)^{12} \\ 0 & \cdots & 0 & 120 & 720(t_l+h) & \cdots & 742560\ \ (t_l+h)^{12} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 120 & 720\left(t_l+\frac{11h}{2}\right) & \cdots & 742560\ \left(t_l+\frac{11h}{2}\right)^{12} \\ 0 & \cdots & 0 & 120 & 720(t_l+6h) & \cdots & 742560\ \ (t_l+6h)^{12} \end{pmatrix}$$

$X = [d_0\ \ d_1\ \ \ldots\ \ d_{17}]^T$

$C = [z_l\ \ z_{l+1}\ \ z_{l+2}\ \ z_{l+3}\ \ z_{l+4}\ \ g_l\ \ g_{l+\frac{1}{2}}\ \ g_{l+1}\ \ g_{l+\frac{3}{2}}\ \ g_{l+2}$
$g_{l+\frac{5}{2}}\ \ g_{l+3}\ \ g_{l+\frac{7}{2}}\ \ g_{l+4}\ \ g_{l+\frac{9}{2}}\ \ g_{l+5}\ \ g_{l+\frac{11}{2}}\ \ g_{l+6}]^T$

Eqn. (9) has been obtained by solving d_k's using inverse method with the help of Matlab code and by substituting in to Eqn. (3), gives a follows:

$$z(t) = \sum_{m=0}^{4} \gamma_m z_{l+m} + \sum_{m=0}^{6} \delta_m g_{l+m} + \sum_{m=0}^{5} \delta_{\nabla_m} g_{l+\nabla m} \qquad (9)$$

By evaluating Eqn. (9) at the non interpolating points $t_{l+\frac{1}{2}}, t_{l+\frac{3}{2}}, t_{l+\frac{5}{2}}, t_{l+\frac{7}{2}}, t_{l+\frac{9}{2}}, t_{l+5}, t_{l+\frac{11}{2}}, t_{l+6}$, yields

$$\left(z_{l+\frac{1}{2}}\quad z_{l+\frac{3}{2}}\quad z_{l+\frac{5}{2}}\quad z_{l+\frac{7}{2}}\quad z_{l+\frac{9}{2}}\quad z_{l+5}\quad z_{l+\frac{11}{2}}\quad z_{l+6}\right)^T$$
$$= \sum_{m=0}^{4} \gamma_m z_{l+m} + \sum_{m=0}^{6} \delta_m g_{l+m} + \sum_{m=0}^{5} \delta_{\nabla_m} g_{l+\nabla m} \qquad (10)$$

Next evaluating the 1st, 2nd, 3rd, and 4th derivatives of Eqn. (9) at each collocation point t_{l+r} respectively, yields

$$z_{l+r}^{(i)} = \sum_{m=0}^{4} \gamma_m^{(i)} z_{l+m} + \sum_{m=0}^{6} \delta_m^{(i)} g_{l+m} + \sum_{m=0}^{5} \delta_{\nabla_m}^{(i)} g_{l+\nabla m} \qquad (11)$$

where $i = 1(1)4$

and

$\nabla_0 = \frac{1}{2},\ \nabla_1 = \frac{3}{2}, \nabla_2 = \frac{5}{2},\ \nabla_3 = \frac{7}{2},\ \nabla_4 = \frac{9}{2},\ \nabla_5 = \frac{11}{2}$

Eqn. (12), Eqn. (13) – Eqn. 16) have been obtained from Eqns. (10) and (11) respectively with the help of Taylor's series expansion on each term.

$$\sum_{\ell=0}^{17} \frac{(ah)^{\ell} z_l^{(\ell)}}{\ell\,!} = \gamma_0 z_l + q_1 + \delta_0 z_l^{(5)} + t_1 \qquad (12)$$

$$z_l' + \sum_{\ell=1}^{16} \frac{(eh)^{\ell} z_l^{(\ell+1)}}{\ell\ !} = \gamma_0' z_l + q_2 + \delta_0' z_l^{(5)} + t_2 \qquad (13)$$

$$z_l'' + \sum_{\ell=1}^{15} \frac{(eh)^{\ell} z_l^{(\ell+2)}}{\ell\ !} = \gamma_0'' z_l + q_3 + \delta_0'' z_l^{(5)} + t_3 \qquad (14)$$

$$z_l''' + \sum_{\ell=1}^{14} \frac{(eh)^{\ell} z_l^{(\ell+3)}}{\ell\ !} = \gamma_0''' z_l + q_4 + \delta_0''' z_l^{(5)} + t_4 \qquad (15)$$

$$z_l^{(4)} + \sum_{\ell=0}^{13} \frac{(eh)^{\ell} z_l^{(\ell+4)}}{\ell\,!} = \gamma^{(4)}{}_0 z_l + q_5 + \delta_0^{(4)} z_l^{(5)} + t_5 \qquad (16)$$

where

$a = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, 5, \frac{11}{2}, 6$

$e = 0(1/2)6$

$$q_1 = \sum_{\tau=1}^{4} \gamma_\tau \left(z_l + \sum_{j=1}^{17} \frac{(\tau h)^j z_l^{(j)}}{j\,!} \right),\quad t_1 = \sum_{\tau=1}^{12} \delta_{\frac{\tau}{2}} \left(z_l^{(5)} + \sum_{u=1}^{12} \frac{\left(\frac{\tau h}{2}\right)^u z_l^{(u+5)}}{u\,!} \right)$$

$$q_2 = \sum_{\tau=1}^{4} \gamma_\tau' \left(z_l + \sum_{j=1}^{17} \frac{(\tau h)^j z_l^{(j)}}{j\,!} \right),\quad t_2 = \sum_{\tau=1}^{12} \delta'_{\frac{\tau}{2}} \left(z_l^{(5)} + \sum_{u=1}^{12} \frac{\left(\frac{\tau h}{2}\right)^u z_l^{(u+5)}}{u\,!} \right)$$

$$q_3 = \sum_{\tau=1}^{4} \gamma_\tau'' \left(z_l + \sum_{j=1}^{17} \frac{(\tau h)^j z_l^{(j)}}{j\,!} \right),\quad t_3 = \sum_{\tau=1}^{12} \delta''_{\frac{\tau}{2}} \left(z_l^{(5)} + \sum_{u=1}^{12} \frac{\left(\frac{\tau h}{2}\right)^u z_l^{(u+5)}}{u\,!} \right)$$

$$q_4 = \sum_{\tau=1}^{4} \gamma'''_\tau \left(z_l + \sum_{j=1}^{17} \frac{(\tau h)^j z_l^{(j)}}{j\,!} \right),\quad t_4 = \sum_{\tau=1}^{12} \delta'''_{\frac{\tau}{2}} \left(z_l^{(5)} + \sum_{u=1}^{12} \frac{\left(\frac{\tau h}{2}\right)^u z_l^{(u+5)}}{u\,!} \right)$$

$$q_5 = \sum_{\tau=1}^{4} \gamma_\tau^{(4)} \left(z_l + \sum_{j=1}^{17} \frac{(\tau h)^j z_l^{(j)}}{j\,!} \right),\quad t_5 = \sum_{\tau=1}^{12} \delta^{(4)}_{\frac{\tau}{2}} \left(z_l^{(5)} + \sum_{u=1}^{12} \frac{\left(\frac{\tau h}{2}\right)^u z_l^{(u+5)}}{u\,!} \right)$$

Rewriting each equation from Eqn. (12) – Eqn. (16) respectively in a matrix form and equating the coefficients of $z_l^{(p_*)}$ for $p^* = 0(1)17$, yields

$$\left.\begin{matrix} HU_0 = E_0 & HU_8 = E_8 & HU_{21} = E_{21} & HU_{34} = E_{34} & HU_{47} = E_{47} \\ HU_1 = E_1 & HU_9 = E_9 & HU_{22} = E_{22} & HU_{35} = E_{35} & HU_{48} = E_{48} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ HU_7 = E_7 & HU_{20} = E_{20} & HU_{33} = E_{33} & HU_{46} = E_{46} & HU_{59} = E_{59} \end{matrix}\right\} \qquad (17)$$

$$U_0 = U_1 = \cdots = U_7 = \left[\gamma_0 \;\; \cdots \;\; \gamma_4 \;\; \delta_r\right]^T$$
$$U_8 = U_9 = \cdots = U_{20} = \left[\gamma_0' \;\; \cdots \;\; \gamma_4' \;\; \delta_r'\right]^T$$
$$U_{21} = U_{22} = \cdots = U_{33} = [\gamma_0'' \;\; \cdots \;\; \gamma_4'' \;\; \delta_r'']^T$$
$$U_{34} = U_{35} = \cdots = U_{46} = [\gamma_0''' \;\; \cdots \;\; \gamma_4''' \;\; \delta_r''']^T$$
$$U_{47} = U_{48} = \cdots = U_{59} = [\gamma_0'''' \;\; \cdots \;\; \gamma_4'''' \;\; \delta_r'''']^T$$

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & h & 2h & 3h & 4h & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{h^2}{2!} & \frac{2^2h^2}{2!} & \frac{(3h)^2}{2!} & \frac{(4h)^2}{2!} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{h^3}{3!} & \frac{2^3h^3}{3!} & \frac{3^3h^3}{3!} & \frac{4^3h^3}{3!} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{h^4}{4!} & \frac{2^4h^4}{4!} & \frac{3^4h^4}{4!} & \frac{4^4h^4}{4!} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{h^5}{5!} & \frac{2^5h^5}{5!} & \frac{3^5h^5}{5!} & \frac{4^5h^5}{5!} & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & \frac{h^6}{6!} & \frac{2^6h^6}{6!} & \frac{3^6h^6}{6!} & \frac{4^6h^6}{6!} & 0 & \frac{h}{2} & h & \frac{3h}{2} & \cdots & \frac{11h}{2} & 6h \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \frac{h^{17}}{17!} & \frac{(2h)^{17}}{17!} & \frac{(3h)^{17}}{17!} & \frac{(4h)^{17}}{17!} & 0 & \frac{\left(\frac{h}{2}\right)^{12}}{12!} & \frac{h^{12}}{12!} & \frac{\left(\frac{3h}{2}\right)^{12}}{12!} & \cdots & \frac{\left(\frac{11h}{2}\right)^{12}}{12!} & \frac{(6h)^{12}}{12!} \end{pmatrix}$$

$$E_0 = \left[1 \;\; \frac{\frac{h}{2}}{1!} \;\; \frac{\left(\frac{h}{2}\right)^2}{2!} \;\; \frac{\left(\frac{h}{2}\right)^3}{3!} \;\; \cdots \;\; \frac{\left(\frac{h}{2}\right)^{17}}{17!}\right]^T \qquad E_4 = \left[1 \;\; \frac{\frac{9h}{2}}{1!} \;\; \frac{\left(\frac{9h}{2}\right)^2}{2!} \;\; \frac{\left(\frac{9h}{2}\right)^3}{3!} \;\; \cdots \;\; \frac{\left(\frac{9h}{2}\right)^{17}}{17!}\right]^T$$

$$E_1 = \left[1 \;\; \frac{\frac{3h}{2}}{1!} \;\; \frac{\left(\frac{3h}{2}\right)^2}{2!} \;\; \frac{\left(\frac{3h}{2}\right)^3}{3!} \;\; \cdots \;\; \frac{\left(\frac{3h}{2}\right)^{17}}{17!}\right]^T \qquad E_5 = \left[1 \;\; \frac{5h}{1!} \;\; \frac{(5h)^2}{2!} \;\; \frac{(5h)^3}{3!} \;\; \cdots \;\; \frac{(5h)^{17}}{17!}\right]$$

$$E_2 = \left[1 \;\; \frac{\frac{5h}{2}}{1!} \;\; \frac{\left(\frac{5h}{2}\right)^2}{2!} \;\; \frac{\left(\frac{5h}{2}\right)^3}{3!} \;\; \cdots \;\; \frac{\left(\frac{h}{2}\right)^{17}}{17!}\right]^T \qquad E_6 = \left[1 \;\; \frac{\frac{11h}{2}}{1!} \;\; \frac{\left(\frac{11h}{2}\right)^2}{2!} \;\; \frac{\left(\frac{11h}{2}\right)^3}{3!} \;\; \cdots \;\; \frac{\left(\frac{11h}{2}\right)^{17}}{17!}\right]^T$$

$$E_3 = \left[1 \;\; \frac{\frac{7h}{2}}{1!} \;\; \frac{\left(\frac{7h}{2}\right)^2}{2!} \;\; \frac{\left(\frac{7h}{2}\right)^3}{3!} \;\; \cdots \;\; \frac{\left(\frac{h}{2}\right)^{17}}{17!}\right]^T \qquad E_7 = \left[1 \;\; \frac{6h}{1!} \;\; \frac{(6h)^2}{2!} \;\; \frac{(6h)^3}{3!} \;\; \cdots \;\; \frac{(6h)^{17}}{17!}\right]^T$$

$$E_8 = \left[0 \;\; 1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0\right]^T \qquad E_{21} = \left[0 \;\; 0 \;\; 1 \;\; 0 \;\; \cdots \;\; 0 \;\; 0\right]^T$$

$$E_9 = \left[0 \;\; 1 \;\; \frac{\frac{h}{2}}{1!} \;\; \frac{\left(\frac{h}{2}\right)^2}{2!} \;\; \cdots \;\; \frac{\left(\frac{h}{2}\right)^{16}}{16!}\right]^T \qquad E_{22} = \left[0 \;\; 0 \;\; 1 \;\; \frac{\frac{h}{2}}{1!} \;\; \frac{\left(\frac{h}{2}\right)^2}{2!} \;\; \cdots \;\; \frac{\left(\frac{h}{2}\right)^{15}}{15!}\right]^T$$

$$E_{10} = \left[0 \;\; 1 \;\; \frac{h}{1!} \;\; \frac{h^2}{2!} \;\; \cdots \;\; \frac{h^{16}}{16!}\right]^T \qquad E_{23} = \left[0 \;\; 0 \;\; 1 \;\; \frac{h}{1!} \;\; \frac{h^2}{2!} \;\; \cdots \;\; \frac{h^{15}}{15!}\right]^T$$
$$\vdots \qquad\qquad\qquad \vdots$$

$$E_{19}=\begin{bmatrix}0 & 1 & \dfrac{\frac{11h}{2}}{1!} & \dfrac{\left(\frac{11h}{2}\right)^2}{2!} & \cdots & \dfrac{\left(\frac{11h}{2}\right)^{16}}{16!}\end{bmatrix}^T \quad E_{32}=\begin{bmatrix}0 & 0 & 1 & \dfrac{\frac{11h}{2}}{1!} & \dfrac{\left(\frac{11h}{2}\right)^2}{2!} & \cdots & \dfrac{\left(\frac{11h}{2}\right)^{15}}{15!}\end{bmatrix}^T$$

$$E_{20}=\begin{bmatrix}0 & 1 & \dfrac{6h}{1!} & \dfrac{(6h)^2}{2!} & \cdots & \dfrac{(6h)^{16}}{16!}\end{bmatrix}^T \quad E_{33}=\begin{bmatrix}0 & 0 & 1 & \dfrac{6h}{1!} & \dfrac{(6h)^2}{2!} & \cdots & \dfrac{(6h)^{15}}{15!}\end{bmatrix}^T$$

$$E_{34}=\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}^T \quad E_{47}=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}^T$$

$$E_{35}=\begin{bmatrix}0 & 0 & 0 & 1 & \dfrac{\frac{h}{2}}{1!} & \dfrac{\left(\frac{h}{2}\right)^2}{2!} & \cdots & \dfrac{\left(\frac{h}{2}\right)^{14}}{14!}\end{bmatrix}^T \quad E_{48}=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & \dfrac{\frac{h}{2}}{1!} & \dfrac{\left(\frac{h}{2}\right)^2}{2!} & \cdots & \dfrac{\left(\frac{h}{2}\right)^{13}}{13!}\end{bmatrix}^T$$

$$E_{36}=\begin{bmatrix}0 & 0 & 0 & 1 & \dfrac{h}{1!} & \dfrac{h^2}{2!} & \cdots & \dfrac{h^{14}}{14!}\end{bmatrix}^T \quad E_{49}=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & \dfrac{h}{1!} & \dfrac{h^2}{2!} & \cdots & \dfrac{h^{13}}{13!}\end{bmatrix}^T$$

$$\vdots \qquad\qquad \vdots$$

$$E_{45}=\begin{bmatrix}0 & 0 & 0 & 1 & \dfrac{\frac{11h}{2}}{1!} & \dfrac{\left(\frac{11h}{2}\right)^2}{2!} & \cdots & \dfrac{\left(\frac{11h}{2}\right)^{14}}{14!}\end{bmatrix}^T \quad E_{58}=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & \dfrac{\frac{11h}{2}}{1!} & \dfrac{\left(\frac{11h}{2}\right)^2}{2!} & \cdots & \dfrac{\left(\frac{11h}{2}\right)^{13}}{13!}\end{bmatrix}^T$$

$$E_{46}=\begin{bmatrix}0 & 0 & 0 & 1 & \dfrac{6h}{1!} & \dfrac{(6h)^2}{2!} & \cdots & \dfrac{(6h)^{14}}{14!}\end{bmatrix}^T \quad E_{59}=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & \dfrac{6h}{1!} & \dfrac{(6h)^2}{2!} & \cdots & \dfrac{(6h)^{13}}{13!}\end{bmatrix}^T$$

The following Tables have been obtained using inverse matrix method in order to find U_0 to U_7 and U_8 to U_{59} from Eqn. (17) and substituting former results into Eqn. (10) and the later results in Eqn. (11) respectively.

Table 1: Evaluation of coefficients γ_m for Eqn. (10) at the non interpolating points.

t	z_l	z_{l+1}	z_{l+2}	z_{l+3}	z_{l+4}
$t_{l+\frac{1}{2}}$	$\frac{35}{128}$	$\frac{35}{32}$	$\frac{-35}{64}$	$\frac{7}{32}$	$\frac{-5}{128}$
$t_{l+3/2}$	$\frac{-5}{128}$	$\frac{15}{32}$	$\frac{45}{64}$	$\frac{-5}{32}$	$\frac{3}{128}$
$t_{l+\frac{5}{2}}$	$\frac{3}{128}$	$\frac{-5}{32}$	$\frac{45}{64}$	$\frac{15}{32}$	$\frac{-5}{128}$
$t_{l+\frac{7}{2}}$	$\frac{-5}{128}$	$\frac{7}{32}$	$\frac{-35}{64}$	$\frac{35}{32}$	$\frac{35}{128}$
$t_{l+\frac{9}{2}}$	$\frac{35}{128}$	$\frac{-45}{32}$	$\frac{189}{64}$	$\frac{-105}{32}$	$\frac{315}{128}$
t_{l+5}	1	-5	10	-10	5
$t_{l+1/2}$	$\frac{315}{128}$	$\frac{-385}{32}$	$\frac{1485}{64}$	$\frac{-693}{32}$	$\frac{1155}{128}$
t_{l+6}	5	-24	45	-45	15

Table 2: Evaluation of coefficients γ'_m for Eqn. (11) at the collocation points.

t	z_l	z_{l+1}	z_{l+2}	z_{l+3}	z_{l+4}
t_l	$\frac{-25}{12h}$	$\frac{4}{h}$	$\frac{-3}{h}$	$\frac{4}{3h}$	$\frac{-1}{4h}$
$t_{l+\frac{1}{2}}$	$\frac{-11}{12h}$	$\frac{17}{24h}$	$\frac{3}{8h}$	$\frac{-5}{24h}$	$\frac{1}{24h}$
t_{l+1}	$\frac{-1}{4h}$	$\frac{-5}{6h}$	$\frac{3}{2h}$	$\frac{-1}{2h}$	$\frac{1}{12h}$
$t_{l+3/2}$	$\frac{1}{24h}$	$\frac{-9}{8h}$	$\frac{9}{8h}$	$\frac{-1}{24h}$	0
t_{l+2}	$\frac{1}{12h}$	$\frac{-2}{3h}$	0	$\frac{2}{3h}$	$\frac{-1}{12h}$
$t_{l+\frac{5}{2}}$	0	$\frac{1}{24h}$	$\frac{-9}{8h}$	$\frac{9}{8h}$	$\frac{-1}{24h}$
t_{l+3}	$\frac{-1}{12h}$	$\frac{1}{2h}$	$\frac{-3}{2h}$	$\frac{5}{6h}$	$\frac{1}{4h}$
$t_{l+\frac{7}{2}}$	$\frac{-1}{24h}$	$\frac{5}{24h}$	$\frac{-3}{8h}$	$\frac{-17}{24h}$	$\frac{11}{12h}$
t_{l+4}	$\frac{1}{4h}$	$\frac{-4}{3h}$	$\frac{3}{h}$	$\frac{-4}{h}$	$\frac{25}{12h}$
$t_{l+\frac{9}{2}}$	$\frac{11}{12h}$	$\frac{-37}{8h}$	$\frac{75}{8h}$	$\frac{-229}{24h}$	$\frac{31}{8h}$
t_{l+5}	$\frac{25}{12h}$	$\frac{-61}{6h}$	$\frac{39}{2h}$	$\frac{-107}{6h}$	$\frac{77}{12h}$
$t_{l+\frac{11}{2}}$	$\frac{31}{8h}$	$\frac{-443}{24h}$	$\frac{273}{8h}$	$\frac{-235}{8h}$	$\frac{59}{6h}$
t_{l+6}	$\frac{77}{12h}$	$\frac{-30}{h}$	$\frac{54}{h}$	$\frac{-134}{3h}$	$\frac{57}{4h}$

Table 3: Evaluation of coefficients γ''_m for Eqn. (11) at the collocation points.

t	z_l	z_{l+1}	z_{l+2}	z_{l+3}	z_{l+4}
t_l	$\frac{35}{12h^2}$	$\frac{-26}{3h^2}$	$\frac{19}{2h^2}$	$\frac{-14}{3h^2}$	$\frac{11}{12h^2}$
$t_{l+\frac{1}{2}}$	$\frac{43}{24h^2}$	$\frac{-14}{3h^2}$	$\frac{17}{4h^2}$	$\frac{-5}{3h^2}$	$\frac{7}{24h^2}$
t_{l+1}	$\frac{11}{12h^2}$	$\frac{-5}{3h^2}$	$\frac{1}{2h^2}$	$\frac{1}{3h^2}$	$\frac{-1}{12h^2}$
$t_{l+3/2}$	$\frac{7}{24h^2}$	$\frac{1}{3h^2}$	$\frac{-7}{4h^2}$	$\frac{4}{3h^2}$	$\frac{-5}{24h^2}$
t_{l+2}	$\frac{-1}{12h^2}$	$\frac{4}{3h^2}$	$\frac{-5}{2h^2}$	$\frac{4}{3h^2}$	$\frac{-1}{12h^2}$
$t_{l+\frac{5}{2}}$	$\frac{-5}{24h^2}$	$\frac{4}{3h^2}$	$\frac{-7}{4h^2}$	$\frac{1}{3h^2}$	$\frac{7}{24h^2}$
t_{l+3}	$\frac{-1}{12h^2}$	$\frac{1}{3h^2}$	$\frac{1}{2h^2}$	$\frac{-5}{3h^2}$	$\frac{11}{12h^2}$
$t_{l+\frac{7}{2}}$	$\frac{7}{24h^2}$	$\frac{-5}{3h^2}$	$\frac{17}{4h^2}$	$\frac{-14}{3h^2}$	$\frac{43}{24h^2}$
t_{l+4}	$\frac{11}{12h^2}$	$\frac{-14}{3h^2}$	$\frac{19}{2h^2}$	$\frac{-26}{3h^2}$	$\frac{35}{12h^2}$
$t_{l+\frac{9}{2}}$	$\frac{43}{24h^2}$	$\frac{-26}{3h^2}$	$\frac{65}{4h^2}$	$\frac{-41}{3h^2}$	$\frac{103}{24h^2}$
t_{l+5}	$\frac{35}{12h^2}$	$\frac{-41}{3h^2}$	$\frac{49}{2h^2}$	$\frac{-59}{3h^2}$	$\frac{71}{12h^2}$
$t_{l+\frac{11}{2}}$	$\frac{103}{24h^2}$	$\frac{-59}{3h^2}$	$\frac{137}{4h^2}$	$\frac{-80}{3h^2}$	$\frac{187}{24h^2}$
t_{l+6}	$\frac{71}{12h^2}$	$\frac{-80}{3h^2}$	$\frac{91}{2h^2}$	$\frac{-104}{3h^2}$	$\frac{119}{12h^2}$

Table 4: Evaluation of coefficients δ_m and δ_{∇_m} for Eqn. (10) at the non interpolating points.

t	g_l	$g_{l+\frac{1}{2}}$	g_{l+1}	$g_{l+\frac{3}{2}}$	g_{l+2}	$g_{l+\frac{5}{2}}$	g_{l+3}	$g_{l+\frac{7}{2}}$	g_{l+4}	$g_{l+\frac{9}{2}}$	g_{l+5}	$g_{l+\frac{11}{2}}$	g_{l+6}
$t_{l+\frac{1}{2}}$	$\frac{-3h^5}{739360}$	$\frac{13h^5}{33571}$	$\frac{80h^5}{18717}$	$\frac{71h^5}{7288}$	$\frac{192h^5}{24385}$	$\frac{270h^5}{58049}$	$\frac{h^5}{3104835}$	$\frac{43h^5}{69163}$	$\frac{-8h^5}{29413}$	$\frac{5h^5}{61347}$	$\frac{-4h^5}{323911}$	$\frac{-h^5}{2887912606}$	$\frac{h^5}{5260374}$
$t_{l+\frac{3}{2}}$	$\frac{h^5}{708521}$	$\frac{-5h^5}{79007}$	$\frac{-59h^5}{77863}$	$\frac{-95h^5}{25851}$	$\frac{-365h^5}{86504}$	$\frac{-66h^5}{23975}$	$\frac{-h^5}{5664027}$	$\frac{-26h^5}{69721}$	$\frac{6h^5}{36943}$	$\frac{-36h^5}{743651}$	$\frac{h^5}{137045}$	$\frac{h^5}{52193015945}$	$\frac{-h^5}{8983274}$
$t_{l+\frac{5}{2}}$	$\frac{-h^5}{500136}$	$\frac{6h^5}{118553}$	$\frac{15h^5}{38393}$	$\frac{35h^5}{13663}$	$\frac{8h^5}{2017}$	$\frac{62h^5}{14297}$	$\frac{h^5}{3709898}$	$\frac{33h^5}{54050}$	$\frac{-20h^5}{76397}$	$\frac{5h^5}{64689}$	$\frac{-2h^5}{172811}$	$\frac{-h^5}{34088376251}$	$\frac{h^5}{5712506}$
$t_{l+\frac{7}{2}}$	$\frac{5h^5}{471926}$	$\frac{-11h^5}{67201}$	$\frac{-34h^5}{132023}$	$\frac{-170h^5}{30959}$	$\frac{-67h^5}{12927}$	$\frac{-226h^5}{16205}$	$\frac{-h^5}{684235}$	$\frac{-51h^5}{15169}$	$\frac{54h^5}{38437}$	$\frac{-3h^5}{7222}$	$\frac{9h^5}{144464}$	$\frac{h^5}{6288364643}$	$\frac{-h^5}{1058857}$
$t_{l+\frac{1}{2}}$	$\frac{-3h^5}{739360}$	$\frac{13h^5}{33571}$	$\frac{80h^5}{18717}$	$\frac{71h^5}{7288}$	$\frac{192h^5}{24385}$	$\frac{270h^5}{58049}$	$\frac{h^5}{3104835}$	$\frac{43h^5}{69163}$	$\frac{-8h^5}{29413}$	$\frac{5h^5}{61347}$	$\frac{-25h^5}{32302}$	$\frac{-h^5}{506669231}$	$\frac{10h^5}{851791}$
$t_{l+\frac{3}{2}}$	$\frac{h^5}{708521}$	$\frac{-5h^5}{79007}$	$\frac{-59h^5}{77863}$	$\frac{-95h^5}{25851}$	$\frac{-365h^5}{86504}$	$\frac{-66h^5}{23975}$	$\frac{-h^5}{5664027}$	$\frac{-26h^5}{69721}$	$\frac{6h^5}{36943}$	$\frac{-36h^5}{743651}$	$\frac{-26h^5}{7919}$	$\frac{-h^5}{119880340}$	$\frac{12h^5}{241513}$
$t_{l+\frac{5}{2}}$	$\frac{-h^5}{500136}$	$\frac{6h^5}{118553}$	$\frac{15h^5}{38393}$	$\frac{35h^5}{13663}$	$\frac{8h^5}{2017}$	$\frac{62h^5}{14297}$	$\frac{h^5}{3709898}$	$\frac{33h^5}{54050}$	$\frac{-20h^5}{76397}$	$\frac{5h^5}{64689}$	$\frac{-14h^5}{1825}$	$\frac{-h^5}{43983981}$	$\frac{20h^5}{148199}$
$t_{l+\frac{7}{2}}$	$\frac{5h^5}{471926}$	$\frac{-11h^5}{67201}$	$\frac{-34h^5}{132023}$	$\frac{-170h^5}{30959}$	$\frac{-67h^5}{12927}$	$\frac{-226h^5}{16205}$	$\frac{-h^5}{684235}$	$\frac{-51h^5}{15169}$	$\frac{54h^5}{38437}$	$\frac{-3h^5}{7222}$	$\frac{-49728h^5}{17}$	$\frac{-61h^5}{3162}$	$\frac{6523h^5}{117}$

Table 5: Evaluation coefficients of δ'_m and δ'_{∇_m} for Eqn. (11) at the collocation point.

t	g_l	$g_{l+\frac{1}{2}}$	g_{l+1}	$g_{l+\frac{3}{2}}$	g_{l+2}	$g_{l+\frac{5}{2}}$	g_{l+3}	$g_{l+\frac{7}{2}}$	g_{l+4}	$g_{l+\frac{9}{2}}$	g_{l+5}	$g_{l+\frac{11}{2}}$	g_{l+6}
t_l	$\frac{3h^4}{192374}$	$\frac{203h^4}{27443}$	$\frac{179h^4}{4464}$	$\frac{380h^4}{5621}$	$\frac{386h^4}{7323}$	$\frac{116h^4}{3945}$	$\frac{h^4}{399369}$	$\frac{61h^4}{14651}$	$\frac{-87h^4}{46145}$	$\frac{30h^4}{52019}$	$\frac{-14h^4}{157907}$	$\frac{-h^4}{3732477430}$	$\frac{h^4}{718957}$
$t_{l+\frac{1}{2}}$	$\frac{6h^4}{489269}$	$-\frac{10h^4}{8039}$	$-\frac{113h^4}{12760}$	$\frac{-82h^4}{6689}$	$-\frac{83h^4}{8844}$	$-\frac{65h^4}{13681}$	$\frac{-h^4}{1479359}$	$-\frac{47h^4}{60930}$	$\frac{17h^4}{45064}$	$-\frac{14h^4}{115867}$	$\frac{10h^4}{520941}$	$\frac{h^4}{14086256892}$	$\frac{-h^4}{3177758}$
t_{l+1}	$\frac{4h^4}{944849}$	$\frac{-8h^4}{22811}$	$\frac{-75h^4}{14744}$	$\frac{-253h^4}{14257}$	$\frac{-161h^4}{10067}$	$-\frac{87h^4}{8744}$	$\frac{-h^4}{1824586}$	$\frac{-53h^4}{41096}$	$\frac{16h^4}{29115}$	$\frac{-23h^4}{142312}$	$\frac{3h^4}{124354}$	$\frac{h^4}{16730473327}$	$\frac{-h^4}{2752972}$
$t_{l+\frac{3}{2}}$	$\frac{h^4}{4334759}$	$\frac{3h^4}{63601}$	$\frac{17h^4}{18397}$	$\frac{57h^4}{20743}$	$\frac{96h^4}{104947}$	$\frac{3h^4}{54109}$	$\frac{-h^4}{57115790}$	$\frac{-h^4}{204383}$	$\frac{3h^4}{704717}$	$\frac{-h^4}{580178}$	$\frac{h^4}{3191710}$	$\frac{h^4}{553488114122}$	$\frac{-h^4}{168097389}$
t_{l+2}	$\frac{-h^4}{215561}$	$\frac{8h^4}{52169}$	$\frac{58h^4}{38201}$	$\frac{77h^4}{8951}$	$\frac{87h^4}{6988}$	$\frac{275h^4}{28264}$	$\frac{h^4}{1680570}$	$\frac{37h^4}{28389}$	$\frac{-62h^4}{110447}$	$\frac{8h^4}{48081}$	$\frac{-h^4}{40016}$	$\frac{-h^4}{15470939062}$	$\frac{h^4}{2634169}$
$t_{l+\frac{5}{2}}$	$\frac{-h^4}{450952}$	$\frac{5h^4}{207383}$	$\frac{-29h^4}{244339}$	$\frac{11h^4}{27623}$	$\frac{5h^4}{21239}$	$\frac{55h^4}{14907}$	$\frac{h^4}{3186131}$	$\frac{25h^4}{36506}$	$\frac{-29h^4}{96650}$	$\frac{22h^4}{246509}$	$\frac{-h^4}{74556}$	$\frac{-h^4}{29262124716}$	$\frac{h^4}{4912666}$
t_{l+3}	$\frac{4h^4}{303479}$	$\frac{-h^4}{4060}$	$\frac{-23h^4}{21704}$	$\frac{-80h^4}{7877}$	$\frac{-158h^4}{11831}$	$\frac{-76h^4}{3375}$	$\frac{-h^4}{555110}$	$\frac{-121h^4}{30566}$	$\frac{27h^4}{15725}$	$\frac{-18h^4}{35315}$	$\frac{2h^4}{26123}$	$\frac{h^4}{5102388521}$	$\frac{-h^4}{860582}$
$t_{l+\frac{7}{2}}$	$\frac{12h^4}{578807}$	$\frac{-53h^4}{191039}$	$\frac{7h^4}{29905}$	$\frac{-133h^4}{17985}$	$\frac{-38h^4}{12289}$	$\frac{-257h^4}{12068}$	$\frac{-h^4}{346365}$	$\frac{-173h^4}{23721}$	$\frac{38h^4}{13421}$	$\frac{-25h^4}{30084}$	$\frac{17h^4}{136878}$	$\frac{h^4}{3181879194}$	$\frac{-h^4}{532819}$
t_{l+4}	$\frac{-11h^4}{114077}$	$\frac{25h^4}{18389}$	$\frac{9h^4}{84898}$	$\frac{188h^4}{4725}$	$\frac{101h^4}{3942}$	$\frac{296h^4}{2745}$	$\frac{5h^4}{373389}$	$\frac{74h^4}{2111}$	$\frac{-243h^4}{18767}$	$\frac{33h^4}{8606}$	$\frac{-19h^4}{33089}$	$\frac{-h^4}{686104411}$	$\frac{5h^4}{575454}$
$t_{l+\frac{9}{2}}$	$\frac{-31h^4}{64407}$	$\frac{161h^4}{25237}$	$\frac{-109h^4}{16712}$	$\frac{472h^4}{2831}$	$\frac{263h^4}{4452}$	$\frac{376h^4}{789}$	$\frac{9h^4}{134465}$	$\frac{565h^4}{2619}$	$\frac{-245h^4}{4534}$	$\frac{55h^4}{2894}$	$\frac{-69h^4}{24200}$	$\frac{-h^4}{137284736}$	$\frac{8h^4}{184845}$
t_{l+5}	$\frac{-22h^4}{17177}$	$\frac{103h^4}{6229}$	$\frac{-151h^4}{6066}$	$\frac{367h^4}{897}$	$\frac{322h^4}{3879}$	$\frac{541h^4}{452}$	$\frac{27h^4}{151228}$	$\frac{2041h^4}{3200}$	$\frac{-245h^4}{2801}$	$\frac{452h^4}{7421}$	$\frac{-62h^4}{8125}$	$\frac{-h^4}{51456347}$	$\frac{17h^4}{146599}$
$t_{l+\frac{11}{2}}$	$\frac{-36h^4}{13703}$	$\frac{145h^4}{4337}$	$\frac{-197h^4}{3306}$	$\frac{592h^4}{739}$	$\frac{180h^4}{2099}$	$\frac{1796h^4}{755}$	$\frac{19h^4}{51917}$	$\frac{1391h^4}{989}$	$\frac{-552h^4}{7645}$	$\frac{377h^4}{1999}$	$\frac{-26h^4}{4609}$	$\frac{-h^4}{25106608}$	$\frac{71h^4}{296921}$
t_{l+6}	$\frac{-158h^4}{27843}$	$\frac{197h^4}{2859}$	$\frac{-191h^4}{1192}$	$\frac{977h^4}{656}$	$\frac{-199h^4}{1750}$	$\frac{239h^4}{56}$	$\frac{40h^4}{44897}$	$\frac{859h^4}{345}$	$\frac{315h^4}{1538}$	$\frac{1236h^4}{3469}$	$\frac{579h^4}{5776}$	$\frac{-h^4}{10125950}$	$\frac{26h^4}{18121}$

Similarly the coefficients γ'''_m γ''''_m δ''_m δ''_{∇_m} δ'''_m δ'''_{∇_m} δ''''_m and δ''''_{∇_m} for Eqn. (11) at the collocation points can be obtained.

The coefficient matrices of Z_R, Z_{R-1}, Z'_{R-1}, Z''_{R-1}, Z'''_{R-1}, Z''''_{R-1} G_{R-1} G_R can be obtained by taking linear combination of all coefficients of Table 1 and 4 with the coefficient from the first row of Table 2, 3. Table 5 and the remain table representations for the coefficients γ'''_m γ''''_m δ''_m δ''_{∇_m} δ'''_m δ'''_{∇_m} δ''''_m and δ''''_{∇_m} respectively. where

$$Z_R = [z_{l+e} \quad z'_{l+e} \quad z''_{l+e} \quad z'''_{l+e} \quad z''''_{l+e}]^T$$

$$Z_{R-1} = [z_{l+t_1}]^T, \quad Z'_{R-1} = [z'_{l+t_1}]^T,$$

$$Z''_{R-1} = [z''_{l+t_1}]^T, \quad Z'''_{R-1} = [z'''_{l+t_1}]^T,$$

$$Z''''_{R-1} = [z''''_{l+t_1}]^T, \quad G_{R-1} = [g_{l+t_1}]^T,$$

$$G_R = [\, g_{l+\frac{1}{2}} \quad g_{l+1} \quad g_{l+\frac{3}{2}} \quad g_{l+2} \quad \cdots \quad g_{l+\frac{29}{2}} \quad g_{l+30} \,]^T$$

$$t_1 = \frac{-59}{2}\left(\frac{1}{2}\right)0$$

Eqn. (18) has been obtained by multiplying each coefficient matrix of

$Z_R,\ Z_{R-1},\ Z'_{R-1},\ Z''_{R-1},\ Z'''_{R-1},\ Z''''_{R-1}\ \ G_{R-1}\ \ G_R$

with the inverse coefficient matrix of

$Z_R = [z_{I+e}\ \ z'_{I+e}\ \ z''_{I+e}\ \ z'''_{I+e}\ \ z''''_{I+e}]^T$.

Further
$$D_0 Z_R = D_1 Z_{R-1} + D_2 Z'_{R-1} + D_3 Z''_{R-1} + D_4 Z'''_{R-1} + D_5 Z''''_{R-1} + D_6 G_{R-1} + D_7 G_R \quad (18)$$

Where D_0 is an identity and D_1, ... , D_7 are matrices of order 60

$$D_1 = \begin{pmatrix} N_1 \\ N_0 \\ N_0 \\ N_0 \\ N_0 \end{pmatrix},\quad D_2 = \begin{pmatrix} hN_2 \\ N_1 \\ N_0 \\ N_0 \\ N_0 \end{pmatrix},\quad D_3 = \begin{pmatrix} h^2N_3 \\ hN_2 \\ N_1 \\ N_0 \\ N_0 \end{pmatrix},\quad D_4 = \begin{pmatrix} h^3N_4 \\ h^2N_3 \\ hN_2 \\ N_1 \\ N_0 \end{pmatrix},$$

$$D_5 = \begin{pmatrix} h^4N_5 \\ h^3N_4 \\ h^2N_3 \\ hN_2 \\ N_1 \end{pmatrix},\quad D_6 = \begin{pmatrix} h^5N_6 \\ h^4N_7 \\ h^3N_8 \\ h^2N_9 \\ hN_{10} \end{pmatrix},\quad D_7 = \begin{pmatrix} h^5N_{11} \\ h^4N_{12} \\ h^3N_{13} \\ h^2N_{14} \\ hN_{15} \end{pmatrix}$$

where N_0, N_1,..., N_{15} are matrices of order 12 × 60 are given by

$N_0 = (d_{nj})$, $d_{nj} = 0$ for all values of n and j ,

$$N_1 = (d_{nj}),\quad d_{nj} = \begin{cases} 1, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$N_2 = (d_{nj}),\quad d_{nj} = \begin{cases} \frac{n}{2}, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$N_3 = (d_{nj}),\quad d_{nj} = \begin{cases} \frac{n^2}{8}, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$N_4 = (d_{nj}),\quad d_{nj} = \begin{cases} \frac{n^3}{48}, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$N_5 = (d_{nj}),\quad d_{nj} = \begin{cases} \frac{n^3}{384}, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$N_6 = (d_{nj}),\quad d_{nj} = \begin{cases} q_1, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$N_7 = (d_{nj}),\quad d_{nj} = \begin{cases} q_2, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$N_8 = (d_{nj}),\quad d_{nj} = \begin{cases} q_3, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$N_9 = (d_{nj}),\quad d_{nj} = \begin{cases} q_4, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$N_{10} = (d_{nj}),\quad d_{nj} = \begin{cases} q_5, & j = 60 \\ 0, & j \neq 60 \end{cases}$$

$$q_1 = \left(\frac{50}{294999}\ \frac{92}{23903}\ \frac{189}{8483}\ \frac{489}{6482}\ \frac{511}{2661}\ \frac{371}{906}\ \frac{3075}{3973}\ \frac{575}{429}\ \frac{2083}{959}\ \frac{558}{167}\ \frac{2011}{408}\ \frac{35055}{4}\right)^T$$

$$q_2 = \left(\frac{63}{40268}\ \frac{70}{4141}\ \frac{280}{4411}\ \frac{13707}{86354}\ \frac{638}{1993}\ \frac{1940}{3433}\ \frac{9276}{10181}\ \frac{1099}{799}\ \frac{2902}{1469}\ \frac{2663}{976}\ \frac{1143}{313}\ \frac{1538}{323}\right)^T$$

$$q_3 = \left(\frac{1586}{142739}\ \frac{108}{1925}\ \frac{309}{2272}\ \frac{455}{1814}\ \frac{2032}{5073}\ \frac{917}{1567}\ \frac{3393}{4217}\ \frac{885}{836}\ \frac{710}{527}\ \frac{583}{349}\ \frac{1221}{602}\ \frac{1374}{569}\right)^T$$

$$q_4 = \left(\frac{265}{4851}\ \frac{394}{3153}\ \frac{464}{2383}\ \frac{4641}{17543}\ \frac{107}{320}\ \frac{997}{2467}\ \frac{864}{1825}\ \frac{528}{973}\ \frac{547}{894}\ \frac{1671}{2453}\ \frac{818}{1093}\ \frac{1994}{2519}\right)^T$$

$$q_5 = \left(\frac{141}{986}\ \frac{269}{1934}\ \frac{185}{1323}\ \frac{1103}{7905}\ \frac{731}{5230}\ \frac{196}{1417}\ \frac{496}{3579}\ \frac{17}{123}\ \frac{227}{1629}\ \frac{322}{2659}\ \frac{686}{12349}\right)^T$$

Since d_{nj} is the n^{th} row and j^{th} column element of the above matrices.

$$N_{11} = \left(\begin{array}{c|c} \begin{array}{c} p_1 \mid p_2 \\ \hline p_3 \mid p_4 \end{array} & p_5 \end{array}\right)$$

where p_5 is a null matrix of order 12 × 48.

$$p_1 = \begin{pmatrix} \frac{13}{54843} & \frac{-21}{52256} & \frac{41}{71822} & \frac{-17}{30850} & \frac{16}{53491} & \frac{-1}{2006776} \\ \frac{44}{4619} & \frac{-386}{28425} & \frac{109}{5761} & \frac{-46}{2541} & \frac{104}{10637} & \frac{-8}{500591} \\ \frac{206}{2869} & \frac{-367}{4277} & \frac{551}{4517} & \frac{-190}{1627} & \frac{358}{5677} & \frac{-14}{136147} \\ \frac{157}{558} & \frac{-635}{2226} & \frac{183}{424} & \frac{-947}{2296} & \frac{1363}{6116} & \frac{-15}{41228} \\ \frac{1123}{1432} & \frac{-368}{535} & \frac{2908}{2553} & \frac{-389}{363} & \frac{714}{1231} & \frac{-19}{20041} \\ \frac{633}{356} & \frac{-1036}{753} & \frac{131}{52} & \frac{-1408}{613} & \frac{2192}{1747} & \frac{-29}{14141} \end{pmatrix}$$

$$p_2 = \begin{pmatrix} \frac{-13}{92845} & \frac{11}{96053} & \frac{-5}{110819} & \frac{2}{246831} & \frac{1}{19820373433} & \frac{-1}{6538724} \\ \frac{-81}{17794} & \frac{49}{13183} & \frac{-93}{63589} & \frac{29}{110525} & \frac{1}{617404323} & \frac{-3}{606676} \\ \frac{-662}{22557} & \frac{271}{11311} & \frac{-11}{1167} & \frac{19}{11237} & \frac{1}{95935731} & \frac{2}{62775} \\ \frac{-201}{1937} & \frac{71}{838} & \frac{-512}{15359} & \frac{55}{9197} & \frac{1}{27115096} & \frac{-10}{88737} \\ \frac{-1368}{5063} & \frac{201}{911} & \frac{-326}{3755} & \frac{69}{4430} & \frac{1}{10406220} & \frac{-11}{37474} \\ \frac{-1087}{1861} & \frac{446}{935} & \frac{-733}{3905} & \frac{365}{10838} & \frac{1}{4810888} & \frac{-11}{17330} \end{pmatrix}$$

$$p_3 = \begin{pmatrix} \frac{1837}{523} & \frac{-629}{258} & \frac{6496}{1317} & \frac{-949}{220} & \frac{2897}{1201} & \frac{-96}{24533} \\ \frac{3201}{509} & \frac{-3896}{981} & \frac{15408}{1745} & \frac{-1653}{224} & \frac{3017}{703} & \frac{-109}{15989} \\ \frac{2835}{271} & \frac{-1746}{287} & \frac{1653}{112} & \frac{-5237}{444} & \frac{2694}{373} & \frac{-29}{2617} \\ \frac{12393}{754} & \frac{-3548}{399} & \frac{4556}{195} & \frac{-5837}{326} & \frac{2081}{179} & \frac{-154}{9027} \\ \frac{8907}{361} & \frac{-1979}{158} & \frac{16667}{471} & \frac{-2949}{113} & \frac{2720}{151} & \frac{-180}{7159} \\ \frac{-178807}{4} & \frac{562861}{5} & -178633 & \frac{545279}{3} & \frac{-508696}{5} & \frac{15152}{79} \end{pmatrix}$$

$$p_4 = \begin{pmatrix} \frac{-305}{274} & \frac{241}{265} & \frac{-354}{989} & \frac{293}{4562} & \frac{1}{2521391} & \frac{-116}{95817} \\ \frac{-1069}{555} & \frac{548}{347} & \frac{-303}{487} & \frac{191}{1710} & \frac{1}{1447426} & \frac{-39}{18514} \\ \frac{-758}{247} & \frac{2999}{1176} & \frac{-779}{774} & \frac{460}{2543} & \frac{1}{890607} & \frac{-73}{21374} \\ \frac{-5281}{1161} & \frac{1475}{378} & \frac{-2093}{1362} & \frac{867}{3131} & \frac{1}{578654} & \frac{-222}{42385} \\ \frac{-22661}{3578} & \frac{3568}{621} & \frac{-2418}{1087} & \frac{502}{1235} & \frac{2}{785507} & \frac{-71}{9242} \\ \frac{197083}{4} & -40730 & \frac{48538}{3} & \frac{-35095}{12} & \frac{-409}{21205} & \frac{7748}{139} \end{pmatrix}$$

Similarly N_{12}, N_{13}, N_{14} and N_{15} can be represented like N_{11}.

Eqn. (19) has been determined from Eqn. (18) and from first row of $D_1, D_2, \ldots, D_7$ each.

$$E_1 = N_1 Z_{R-1} + hN_2 Z'_{R-1} + h^2 N_3 Z''_{R-1} + h^3 N_4 Z'''_{R-1} + h^4 N_5 Z^{(iv)}_{R-1} Z_{R-1} + h^5 N_6 G_{R-1} + h^5 N_{11} G_R \quad (19)$$

Similarly we can obtained the corresponding values of $(z'_{l+e})^T$, $(z''_{l+e})^T$, $(z'''_{l+e})^T$, $(z''''_{l+e})^T$ and $(z'''''_{l+e})^T$ as follows:

$$(z'_{l+e})^T = N_0 Z_{R-\partial} + N_1 Z'_{R-\partial} + N_2 Z''_{R-\partial} + N_3 Z'''_{R-\partial} + N_4 Z''''_{R-\partial} + N_7 G_{R-\partial} + N_{12} G_R \quad (20a)$$

$$(z''_{l+e})^T = N_0 Z_{R-\partial} + N_0 Z'_{R-\partial} + N_1 Z''_{R-\partial} + N_2 Z'''_{R-\partial} + N_3 Z^{(iv)}{}_{R-\partial} + N_8 G_{R-\partial} + N_{13} G_R \quad (20b)$$

$$(z'''_{l+e})^T = N_0 Z_{R-\partial} + N_0 Z'_{R-\partial} + N_0 Z''_{R-\partial} + N_1 Z'''_{R-\partial} + N_2 Z^{(iv)}{}_{R-\partial} + N_9 G_{R-\partial\lambda} + N_{14} G_R \quad (20c)$$

$$(z''''_{l+e})^T = N_0 Z_{R-\partial} + N_0 Z'_{R-\partial} + N_0 Z''_{R-\partial} + N_0 Z'''_{R-\partial} + N_1 Z^{(iv)}{}_{R-\partial} + N_{10} G_{R-\partial} + N_{15} G_R \quad (20d)$$

III. ESSENTIAL PROPERTIES OF HYBRID BLOCK APPROACH

To find the essential properties of the scheme rewriting Eqn. (19) in the form

$$M_0 Z_U = M_1 Z_{U-1} + hM_2 Z'_{U-1} + h^2 M_3 Z''_{U-1} + h^3 M_4 Z'''_{U-1} + h^4 M_5 Z''''_{U-1} + h^5 M_6 G_{U-1} + h^5 M_7 G_U = \sum_{i=0}^{\alpha-1} h^i M_{i+1} Y^{(i)}_{U-1} + h^{\alpha} \sum_{i=0}^{1} M_{7-i} G_{U-i} \quad (21)$$

M_0 is an identity matrix and M_1, M_2, ..., M_7 are coefficient matrices of order 12 × 12, yields

$$M_1 = (a_{n_1 m_1}), \quad a_{n_1 m_1} = \begin{cases} 1, & m_1 = 12 \\ 0, & m_1 \neq 12 \end{cases}$$

$$M_2 = (a_{n_1 m_1}), \quad a_{n_1 m_1} = \begin{cases} \frac{n_1}{2}, & m_1 = 12 \\ 0, & m_1 \neq 12 \end{cases}$$

$$M_3 = (a_{n_1 m_1}), \quad a_{n_1 m_1} = \begin{cases} \frac{n_1^2}{8}, & m_1 = 12 \\ 0, & m_1 \neq 12 \end{cases}$$

$$M_4 = (a_{n_1 m_1}), \quad a_{n_1 m_1} = \begin{cases} \frac{n_1^3}{48}, & m_1 = 12 \\ 0, & m_1 \neq 12 \end{cases}$$

$$M_5 = (a_{n_1 m_1}), \quad a_{n_1 m_1} = \begin{cases} \frac{n_1^4}{384}, & m_1 = 12 \\ 0, & m_1 \neq 12 \end{cases}$$

$$M_6 = (a_{n_1 m_1}),$$
$$a_{n_1 m_1} = \begin{cases} \left(\frac{50}{294999} \; \frac{92}{23903} \; \frac{189}{8483} \; \frac{489}{6482} \; \frac{511}{2661} \; \frac{371}{906} \; \frac{3075}{3973} \; \frac{575}{429} \; \frac{2083}{959} \; \frac{558}{167} \; \frac{2011}{408} \; \frac{35055}{4} \right)^T, & m_1 = 12 \\ 0, & m_1 \neq 12 \end{cases}$$

$$M_7 = \begin{pmatrix} p_1 \mid p_2 \\ p_3 \mid p_4 \end{pmatrix}$$

$$Z_U = \left(z_{l+\frac{1}{2}} \;\; z_{l+1} \;\; z_{l+\frac{3}{2}} \cdots z_{l+\frac{11}{2}} \;\; z_{l+6} \right)^T$$

$$Z_{U-1} = \left(z_{l-\frac{11}{2}} \;\; z_{l-5} \;\; z_{l-\frac{9}{2}} \cdots z_{l-\frac{1}{2}} \;\; z_l \right)^T$$

$$Z'_{U-1} = \left(z'_{l-\frac{11}{2}} \;\; z'_{l-5} \;\; z'_{l-\frac{9}{2}} \cdots z'_{l-\frac{1}{2}} \;\; z'_l \right)^T$$

$$Z''_{U-1} = \left(z''_{l-\frac{11}{2}} \;\; z''_{l-5} \;\; z''_{l-\frac{9}{2}} \cdots z''_{l-\frac{1}{2}} \;\; z''_l \right)^T$$

$$Z'''_{U-1} = \left(z'''_{l-\frac{11}{2}} \;\; z'''_{l-5} \;\; z'''_{l-\frac{9}{2}} \cdots z'''_{l-\frac{1}{2}} \;\; z'''_l \right)^T$$

$$Z''''_{U-1} = \left(z''''_{l-\frac{11}{2}} \;\; z''''_{l-5} \;\; z''''_{l-\frac{9}{2}} \cdots z''''_{l-\frac{1}{2}} \;\; z''''_l \right)^T$$

$$G_{U-1} = \left(g_{l-\frac{11}{2}} \;\; g_{l-5} \;\; g_{l-\frac{9}{2}} \cdots g_{l-\frac{1}{2}} \;\; g_l \right)^T$$

$$G_U = \left(g_{l+\frac{1}{2}} \;\; g_{l+1} \;\; g_{l+\frac{3}{2}} \cdots g_{l+\frac{11}{2}} \;\; g_{l+6} \right)^T$$

A. Order of Hybrid Block Method

The linear operator associated with Eqn. (21) has represented in terms of Eqn. (22).

$$L(z(t):h) = M_0 Z_U - M_1 Z_{U-1} - hM_2 Z'_{U-1} - h^2 M_3 Z''_{U-1} - h^3 M_4 Z'''_{U-1} - h^4 M_5 Z''''_{U-1} - h^5 M_6 G_{U-1} - h^5 M_7 G_U \quad (22)$$

Eqn. (23) has been obtained from the Taylor series expansion of Eqn. (22)

$$L(z(t):h) = \varepsilon_0 z(t) + \varepsilon_1 hz'(t) + \varepsilon_2 h^2 z''(t) + \ldots + \varepsilon_{\phi} h^{\phi} z^{(\phi)}(t) + \varepsilon_{\phi+1} h^{\phi+1} z^{(\phi+1)}(t) + \varepsilon_{\phi+2} h^{\phi+2} z^{(\phi+2)}(t) + \ldots \quad (23)$$

If $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{\phi} = \varepsilon_{\phi+1} = 0$ and $\varepsilon_{\phi+2} \neq 0$,

then $\varepsilon_{\phi+2}$ is the error constant with the local truncation error $\xi_{n+k} = \varepsilon_{\phi+2} h^{\phi+2} y^{(\phi+2)}(x) + O(h^{\phi+3})$, then ϕ is said to be order of the block method [26, 27]. Then the six step hybrid block method has order 12 with $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{13} = 0$ and error constant

$$\varepsilon_{14} = \left[\frac{-1}{233997} \; \frac{1}{50397147} \; \frac{1}{7703650} \; \frac{1}{1993498} \; \frac{1}{770112} \; \frac{3}{1182176} \; \frac{3}{534913} \; \frac{2}{274455} \; \frac{1}{1777827} \; \frac{-1}{567060} \; \frac{-2011}{408} \; \frac{125}{9106} \right]^T$$

B. Consistent and zero Stability of the Method

Zero stability of the method:

Since, $P(\lambda) = \det(\lambda M_0 - M_1) = 0$

From Eqn. (21), M_0 and M_1 are the coefficients of $z_{l+\tau}, \tau = \frac{1}{2}\left(\frac{1}{2}\right)6$ and z_l

$\Rightarrow P(\lambda) = \left| \lambda M_0 - M_1 \right| = 0$

$\Rightarrow P(\lambda) = \lambda^{11}(\lambda - 1) = 0$

$\Rightarrow \lambda = 0,0,0,0,0,0,0,0,0,0,0,1$

Therefore the hybrid block method is zero stable [4].

Consistent of the method:

The given method is said to be consistent if it has order $\varepsilon \geq 1$

C. Convergence

The present method is convergent as it is zero stable and has order greater than one [3].

D. Interval of absolute stability of scheme

The region of absolute stability of the block method for boundary locus [18, 24] has been obtained with $\bar{y}(\lambda) = \frac{\psi(\lambda)}{\upsilon(\lambda)}$, where $\psi(\lambda)$ and $\upsilon(\lambda)$ are the first and second characteristics polynomial respectively [5].

Substituting the set problem of the form $z^{(\alpha)} = v^{\alpha} z$ in Eqn. (21), yields.

$$M_0 Z_J = \sum_{i=0}^{\alpha-1} h^i M_{i+1} Z_{U-1}^{(i)} + h^{\alpha} \sum_{i=0}^{1} v^{\alpha} \quad M_{7-i} z_{U-i} \tag{24}$$

which gives

$$\bar{y}(\lambda, h) = \frac{C_0 Z_U(\lambda) - C_1 Z_{U-1}(\lambda)}{C_7 z_U(\lambda) + C_6 z_{U-1}(\lambda)} \tag{25}$$

where $\bar{y} = v^{\alpha} h^{\alpha}$.

Eqn. (25) is written in Euler's form as

$$\bar{y}(\theta, h) = \frac{C_0 Z_H(\theta) - C_1 Z_{H-1}(\theta)}{C_7 z_H(\theta) + C_6 z_{H-1}(\theta)}, \text{ where } \lambda = e^{i\theta} \tag{26}$$

Eqn. (26) is called the characteristics matrix.

Using Eqn. (26) to the new six step hybrid block method, stated in Eqn. (21), yields

$$\bar{y}(\theta, h) = \frac{Q - M_1}{J + M_6} \tag{27}$$

$$Q = M_0 \left(e^{\frac{i}{2}\theta} \quad e^{i\theta} \quad e^{\frac{3i}{2}\theta} \quad e^{2i\theta} \quad e^{\frac{5i}{2}\theta} \quad e^{3i\theta} \quad e^{\frac{7i}{2}\theta} \quad e^{4i\theta} \quad e^{\frac{9i}{2}\theta} \quad e^{5i\theta} \quad e^{\frac{11i}{2}\theta} \quad e^{6i\theta} \right)^T$$

$$J = M_7 \left(e^{\frac{i}{2}\theta} \quad e^{i\theta} \quad e^{\frac{3i}{2}\theta} \quad e^{2i\theta} \quad e^{\frac{5i}{2}\theta} \quad e^{3i\theta} \quad e^{\frac{7i}{2}\theta} \quad e^{4i\theta} \quad e^{\frac{9i}{2}\theta} \quad e^{5i\theta} \quad e^{\frac{11i}{2}\theta} \quad e^{6i\theta} \right)^T$$

where the values of M_1, M_6 and M_7 are the same with in Eqn. (21).

Obtaining the determinants of Eqn. (27) and simplifying it

$$\bar{y}(\theta, h) = \frac{1.264008916528996 \times 10^{35}(e^{6i\theta} - 1)}{6.742376624623854 \times 10^{38} e^{6i\theta} + 7.70587234134904 \times 10^{38}} \tag{28}$$

Expressing Eqn. (28) in trigonometric form and considering it's real part only, yields

$$\bar{y}(\theta, h) = \frac{1.264008916528996 \times 10^{35}(\cos(6\theta) - 1)}{6.742376624623854 \times 10^{38}\cos(6\theta) + 7.70587234134904 \times 10^{38}}$$

The results of Eqn. (29) at intervals of θ of 30° which is shown in the Table 6.

Table 6: Interval of absolute stability.

θ	0°	30°	60°	90°	120°	150°	180°
$\bar{y}(\theta, h)$	0	− 0.13063	0	− 0.13063	0	− 0.13063	0

Form Table 6, we have observed that the interval of absolute stability is given by $(-0.13063, 0)$ and since cosine function is periodic then the interval of periodicity lies in the interval $(-\infty, 0)$.

IV. NUMERICAL VERIFICATION

For numerical verification of the proposed method three test problems have been considered.

1. $z^{(v)} = 2z'z'' - zz^{(iv)} - z'z''' - 8t + (t^2 - 2t - 3)e^t$,
   $0 \leq t \leq 1,\ h = 0.1,\ z(0) = z'(0) = 1,\ z''(0) = 3,$
   $z'''(0) = z''''(0) = 1$. Exact solution $z(t) = t^2 + e^t$.
2. $z^{(5)} = -(t-2)z^{(4)} - 2z^{(3)} + (t^2+2t-1)z'' - (2t^2+4t)z'$ $+2t^2 z + 4e^t \cos(t) - 2t^4 + 4t^3 + 6t^2 - 4t + 2,$
   $0 \leq t \leq 1,\ h = 0.1,\ z(0) = 0,\ z'(0) = 2,$
   $z''(0) = 6,\ z'''(0) = 4,\ z^{(4)}(0) = 0.$

   Exact solution $z(t) = 2e^t \sin(t) + t^2$.
3. $z^{(5)} - z' = t,\ 0 \leq t \leq 1,\ z(0) = z'(0) = 1,$
   $z''(0) = z'''(0) = 0,\ z''''(0) = -1$.

   Exact solution

   $$z(t) = 2 + \frac{1}{2}\sinh(t) - \cos(t) - \frac{1}{2}\left(-\sin(t) + t^2\right)$$

   with $h = 0.05$

Table 7: Result analysis of Example 1.

t	Exact	Numerical	Error in [23] with h = 0.01	Error in [2] with h = 0.01	Error in ODE 45	Error in the current method
0.1	1.115170918075648	1.115170918075648	1.459721 e^{-12}	1.210563e^{-8}	4.7269e^{-9}	2.0748e^{-17}
0.2	1.261402758160170	1.261402758160170	4.187584e^{-11}	1.927066e^{-8}	5.5036e^{-9}	5.6269e^{-17}
0.3	1.439858807576003	1.439858807576003	3.221776 e^{-10}	3.025973e^{-8}	6.4259e^{-9}	5.6056e^{-17}
0.4	1.651824697641271	1.651824697641270	1.365175e^{-9}	4.697311e^{-8}	7.5233e^{-9}	1.7181e^{-16}
0.5	1.898721270700128	1.898721270700128	4.166737e^{-9}	7.208283e^{-8}	8.8279e^{-9}	4.7338e^{-17}
0.6	2.182118800390509	2.182118800390509	1.033164e^{-8}	1.092800e^{-7}	1.0374e^{-8}	8.9512e^{-18}
0.7	2.503752707470476	2.503752707470476	2.218906e^{-8}	1.635841e^{-7}	1.2200e^{-8}	2.0311e^{-16}
0.8	2.865540928492468	2.865540928492468	4.288554e^{-8}	2.417328e^{-7}	1.4346e^{-8}	3.9329e^{-16}
0.9	3.269603111156950	3.269603111156950	7.645583e^{-8}	3.526740e^{-7}	1.6858e^{-8}	7.0203e^{-17}
1	3.718281828459046	3.718281828459045	1.278750e^{-8}	5.081954e^{-7}	8.1947e^{-9}	2.9953e^{-16}

Table 8: Result analysis of Example 2.

t	Exact	Numerical	Error in [1]	Error in ODE45	Error in the current method
0.1	0.230665977460407	0.230665977460407	$5.7e^{-12}$	$1.3681e^{-9}$	$3.2982e^{-17}$
0.2	0.525310537189846	0.525310537189846	$9.7e^{-11}$	$6.1350e^{-9}$	$1.2476e^{-17}$
0.3	0.887821107556980	0.887821107556980	$5.3e^{-10}$	$1.1788e^{-8}$	$7.1918e^{-17}$
0.4	1.321887801541135	1.321887801541134	$1.7e^{-9}$	$1.8222e^{-8}$	$1.6211e^{-16}$
0.5	1.830878166427230	1.830878166427230	$4.5e^{-9}$	$2.5254e^{-8}$	$3.1412e^{-17}$
0.6	2.417691332544183	2.417691332544183	$1.0e^{-8}$	$3.2604e^{-8}$	$5.8104e^{-17}$
0.7	3.084590223750538	3.084590223750538	$1.91e^{-8}$	$3.9882e^{-8}$	$4.1569e^{-16}$
0.8	3.833010681200503	3.833010681200502	$3.51e^{-8}$	$4.6573e^{-8}$	$7.2580e^{-16}$
0.9	4.663346607945434	4.663346607945434	$4.6633e^{-8}$	$5.2082e^{-8}$	$4.3387e^{-16}$
1	5.574710574357686	5.574710574357685	Not define	$5.5084e^{-8}$	$6.8017e^{-16}$

Table 9: Result analysis of Example 3.

t	Exact	Numerical	Error in ODE 45	Error in the current method
0.05	1.049999742209200	1.049999742209200	$2.4193e^{-9}$	$1.2822e^{-17}$
0.1	1.099995918055310	1.099995918055311	$2.9470e^{-9}$	$4.4423e^{-16}$
0.15	1.149979554876563	1.149979554876563	$2.95004e^{-9}$	$8.2074e^{-17}$
0.2	1.199936088826836	1.199936088826836	$3.54135e^{-9}$	$2.9946e^{-17}$
0.25	1.249845716320701	1.249845716320701	$3.4775e^{-9}$	$1.5621e^{-17}$
0.3	1.299683760928635	1.299683760928635	$4.12987e^{-9}$	$3.3635e^{-16}$
0.35	1.349421055598983	1.349421055598983	$3.99182e^{-9}$	$4.2207e^{-17}$
0.4	1.399024340052848	1.399024340052848	$4.70243e^{-9}$	$2.1373e^{-16}$
0.45	1.448456673170037	1.448456673170037	$4.48473e^{-9}$	$3.1379e^{-16}$
0.5	1.497677860158603	1.497677860158603	$5.2508e^{-9}$	$2.7104e^{-17}$
0.55	1.546644894277551	1.546644894277551	$4.94945e^{-9}$	$1.4070e^{-17}$
0.6	1.595312412861960	1.595312412861960	$5.76839e^{-9}$	$2.2171e^{-16}$
0.65	1.643633167382184	1.643633167382184	$5.38130e^{-9}$	$8.3097e^{-17}$
0.7	1.691558507254124	1.691558507254124	$6.25049e^{-9}$	$5.0866e^{-17}$
0.75	1.739038877105761	1.739038877105761	$5.77756e^{-9}$	$2.8662e^{-17}$
0.8	1.786024327196408	1.786024327196407	$6.69482e^{-9}$	$3.3825e^{-16}$
0.85	1.832465036679480	1.832465036679480	$6.1375e^{-9}$	$4.8288e^{-16}$
0.9	1.878311849397165	1.878311849397165	$7.1011e^{-9}$	$1.1678e^{-16}$
0.95	1.923516821896140	1.923516821896140	$3.0720e^{-9}$	$1.9654e^{-16}$
1	1.968033783357709	1.968033783357710	$1.94665e^{-9}$	$1.98776e^{-16}$

The maximum absolute error of the new method and ODE45 are $4.8288e^{-16}$, $7.1011e-09$, respectively.

The notation: ODE45 is Runge-Kutta dormand prince ordinary differential equation solver.

V. CONCLUSION

We observed that the approximate solution obtained by the six-step hybrid block method is more accurate and has good agreement with analytical solutions. The numerical evidence as reported in Table 7, 8, 9 shows that the present method is comfortable to obtain the solution of any general fifth order initial value problems. Since our method is 12th order it has been observed that the proposed method far suitable than the result obtained with ODE113 rather than ODE45. The functional evaluation and computing time is less expensive in comparison to standard multistep methods.

VI. FUTURE WORK

The present work may be extended for the approximate solution of initial value problems of higher order ODE.

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How to cite this article: Gebremedhin, Guesh Simretab and Jena, Saumya Ranjan (2019). Approximate Solution of Ordinary Differential Equations via Hybrid Block Approach. *International Journal on Emerging Technologies*, *10*(4): 201–211.